Foundations

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Contents

1 Vectors

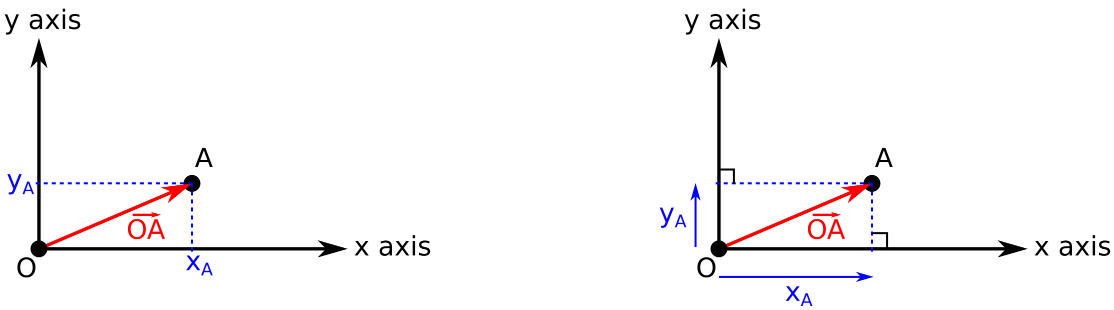
This course is about the laws of motion. First, we need a mathematical system to describe where an object is. Then, we can discuss its motion.

1.1 Cartesian coordinates

Cartesian coordinates, also known as rectangular coordinates, are the standard way to communicate the location of a point in math. A cartesian coordinate system consists of an origin, usually called O, and two axes, usually called x and y. The x and y coordinates of a point A, usually called x_A and y_A, are the lengths you need to travel along each axis (x_A along the x axis and y_A along the y axis) to go from the origin O to point A.

If you already have A drawn on a sketch, you can find its coordinates x_A and y_A by projecting point A orthogonally onto the x and y axes, respectively.

Both axes are oriented, meaning they have a positive direction (indicated by an arrow) and a negative direction. If going from O to A requires traveling along the x axis in the negative direction (opposite the x-axis arrow), then x_A is negative. Similarly, if you need to travel along the y axis in the negative direction (opposite the y-axis arrow), then y_A is negative.



Note: Although x usually points to the right and y usually points up, this will not always be the case. The definitions above are carefully worded to remain true regardless of how tilted the axes are. For example, the statement "x_A is negative if A is left of the y axis" is correct for the sketch above, but would be incorrect if x was pointing up and y was pointed left.

1.2 Position vectors

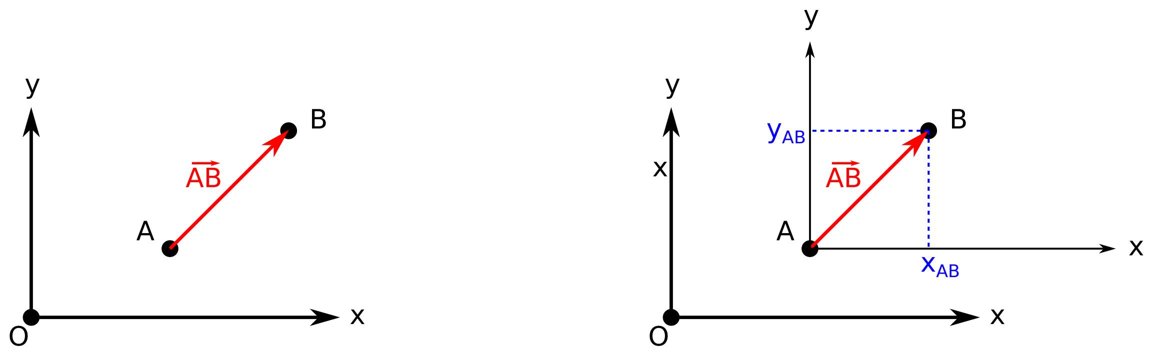
The red arrow going from the origin O to point A is the *position* vector of point A, usually noted $\overrightarrow{OA}$ or $\vec{r}_A$. The vector has an x component, which is the same thing as the point's x coordinate, and a y component, which is the same thing as the point's y coordinate. The vector's components are often written in column format with either square brackets or parentheses:

$$\overrightarrow{OA} = \begin{bmatrix} x_A \\ y_A \end{bmatrix} \text{ or } \overrightarrow{OA} = \begin{pmatrix} x_A \\ y_A \end{pmatrix} \text{ or } \vec{r}_A = \begin{bmatrix} x_A \\ y_A \end{bmatrix} \text{ or } \vec{r}_A = \begin{pmatrix} x_A \\ y_A \end{pmatrix}.$$

For example, $\overrightarrow{OA} = \begin{bmatrix} 3\,\text{cm} \\ 2\,\text{cm} \end{bmatrix}$ means that if you start at O, travel 3 cm in the direction of the x axis, then 2 cm in the direction of the y axis, you'll arrive at A.

1.3 Relative position vectors

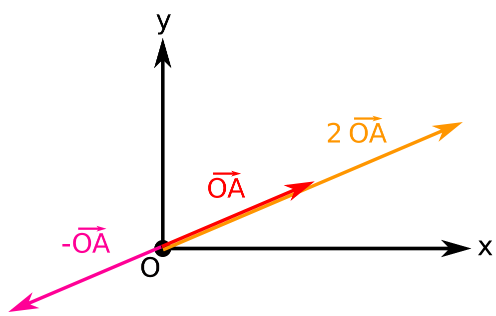
Sometimes we need to need to specify the location of a point relative to another point which is not the origin. The arrow pointing from point A to point B is called the position of B relative to A, noted $\overrightarrow{AB}$. Its components, called x_{AB} and y_{AB}, are defined the same way as before except we temporarily treat A as the origin.



If we're discussing an object moving from A to B, then $\overrightarrow{AB}$ may also be called the object's *displacement vector*.

1.4 Multiplying vectors

A vector can be multiplied by a number. The result is a new vector with the same direction as the original vector, but a new length which is the old length times the number.



In terms of components, multiplying a vector by a number means multiplying every component by that number:

$$2\,\overrightarrow{OA} = 2\begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} 2x_A \\ 2y_A \end{bmatrix}$$

$$-\overrightarrow{OA} = -\begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} -x_A \\ -y_A \end{bmatrix}$$

If you look at the graph, that makes sense. To go from O to the end of $2\overrightarrow{OA}$, you need to travel twice as far along the x axis and twice as far along the y axis. To go from O to the end of $-\overrightarrow{OA}$, you need to travel the same length along each axis as for $\overrightarrow{OA}$, but in the opposite direction, so the components change sign.

Problem 1: Vector multiplication.

If $\overrightarrow{OA}$ has components $\begin{bmatrix} x_A \\ y_A \end{bmatrix}$, what are the components of $\overrightarrow{AO}$? What is the mathematical relationship between $\overrightarrow{OA}$ and $\overrightarrow{AO}$?

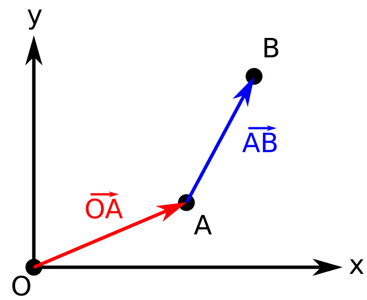
1.5 Adding vectors

To add vectors, you add their components separately (x components together, then y components together):

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

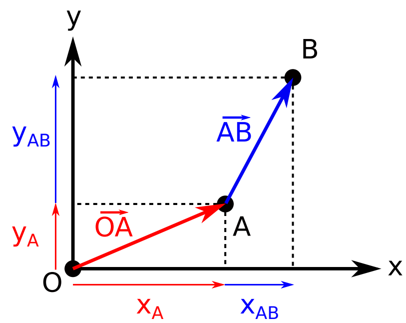
A simple example

To see what this means graphically, let's consider two points A and B (three with the origin O).



The sum of $\overrightarrow{OA}$ and $\overrightarrow{AB}$ is $\overrightarrow{OA}+\overrightarrow{AB} = \begin{bmatrix} x_A \\ y_A \end{bmatrix} + \begin{bmatrix} x_{AB} \\ y_{AB} \end{bmatrix} = \begin{bmatrix} x_A + x_{AB} \\ y_A + y_{AB} \end{bmatrix}$.

The sketch below shows where we can visualize each component:

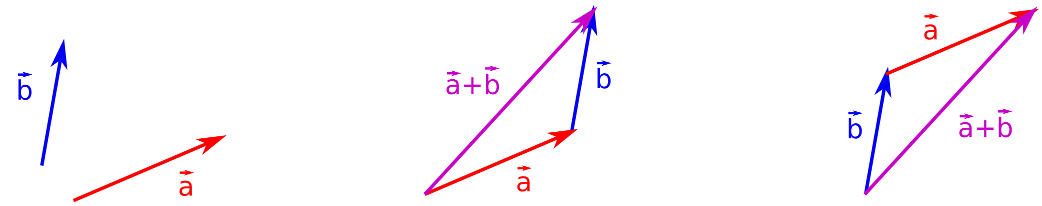


Problem 2: Vector addition.

1. What arrows can we draw to visualize $x_A + x_{AB}$ and $y_A + y_{AB}$? Draw the position vector (the arrow representing it) whose components are $x_A + x_{AB}$ and $y_A + y_{AB}$. Summarize this result as a vector equation involving O, A, and B.
2. We're just one step away from the formulas for the components of a relative position vector (x_{AB} and y_{AB}) in terms of the components of the two points coordinates (x_A, x_B, y_A, y_B). What are those formulas?

General case

In the example above the two vectors are "connected": the tip of the first vector is the base of the other. However, vector addition is not limited to this situation. The addition formula (adding the components separately) works for any two vectors. To perform the addition graphically, simply translate one vector until its base touches the tip of the other vector. Then you're back to the case we already discussed.



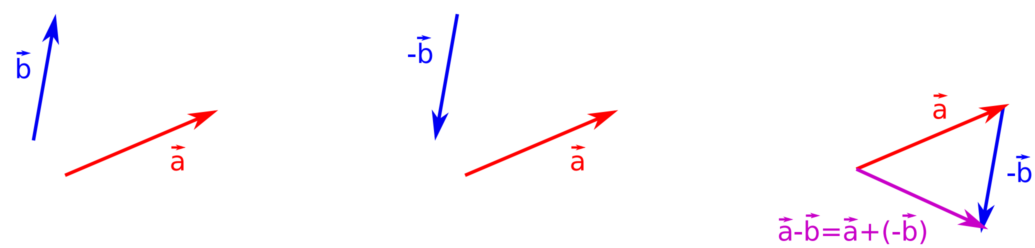
The order of the addition ($\vec{a}+\vec{b}$ vs $\vec{b}+\vec{a}$) doesn't matter. This is obvious from the component formula:

$$\vec{a}+\vec{b} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \end{bmatrix} = \begin{bmatrix} b_x + a_x \\ b_y + a_y \end{bmatrix} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} + \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \vec{b}+\vec{a}$$

where a_x, a_y, b_x, b_y are the components of $\vec{a}$ and $\vec{b}$.

Vector subtraction

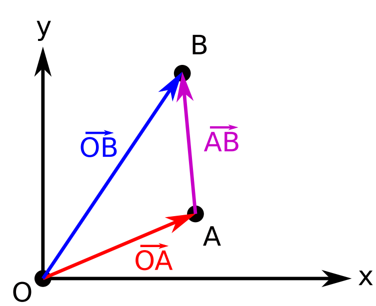
A subtraction is just a multiplication by -1 followed by an addition:



Like normal (number) subtractions, this time the order matters.

Back to relative position vectors

In my view, the easiest way to remember the formula for the relative position is to frame it as a graphical subtraction:



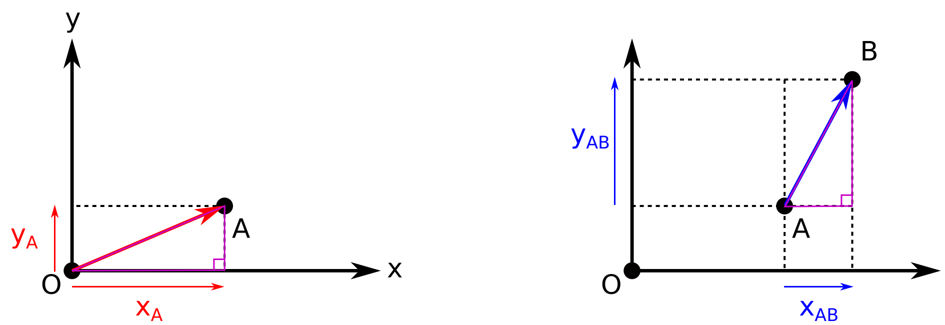
$\overrightarrow{AB}$ is the vector going from A to B. We want the components of $\overrightarrow{AB}$ as a function of the coordinates of A and B. Since those coordinates are measured from the origin, we need to involve the origin. More specifically, we want to write $\overrightarrow{AB}$ as a function of the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$. To do that, we add a detour to $\overrightarrow{AB}$: we go from A, to O, to B. The vector equality corresponding to that detour is $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$. The first vector on the right-hand side is not quite the position vector of A: instead of going from O to A, it goes from A to O, i.e., its the same trip in reverse. To get the correct direction, we need to flip it with a minus sign: $\overrightarrow{AO} = -\overrightarrow{OA}$. Plugging that into the detour equation yields $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, which is the formula for the relative position as a function of the position vectors of the two points. To get the component formulas, we replace the vectors by their components:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \Longrightarrow \begin{bmatrix} x_{AB} \\ y_{AB} \end{bmatrix} = \begin{bmatrix} x_B \\ y_B \end{bmatrix} - \begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix} \Longrightarrow x_{AB} = x_B - x_A \text{ and } y_{AB} = y_B - y_A$$

1.6 Magnitude and distance

The magnitude of a vector is the "length" of the arrow representing the vector. Try not to use the word "length" though, because later we'll compute the magnitudes that are not lengths but speeds, forces, etc, and using "length" in those cases is confusing.

The magnitude of a position vector is also the distance between that point and the origin, and the magnitude of a relative position vector is the distance between the two points. Either way, the magnitude can be computed from the vector's components using the Pythagorean theorem:



The dashed lines between the points and their coordinates represent orthogonal projections, therefore the purple triangles are right triangles and we can apply the Pythagorean theorem:

$$\|\overrightarrow{OA}\| = \sqrt{x_A^2 + y_A^2}, \quad \|\overrightarrow{AB}\| = \sqrt{x_{AB}^2 + y_{AB}^2} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

This is not limited to vectors involving positions. Later in the course we'll compute the magnitude of velocity vectors, force vectors, etc. The formula remains the same: the magnitude is the square root of the sum of the squares of the components.

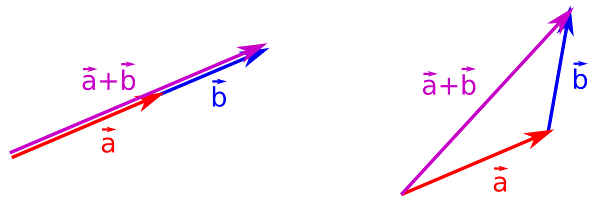
Note: Squares are always positive, so a sum of squares is always positive as well, and the square root of a positive number is a positive number, so *the magnitude is always positive.* That makes sense if we think of it as a length or distance. In the context of the distance between two points A and B, this matches the intuition that the distance between A and B is the same thing as the distance between B and A.

Magnitude, multiplication, and addition

If you multiply a vector by a number, its magnitude gets multiplied by the absolute value of that number. That makes sense in light of the note above. Multiplying a vector by -2 means multiplying it by -1, which reverses its direction without changing its size, then multiplying it by 2, which doubles its size. Therefore, multiplying the vector by -2 multiplies its magnitude by 2.

Division works the same: dividing a vector by a number divides its magnitude by the absolute value of that number. It really can't work any other way if you think about the fact that dividing a vector by a number α is really just multiplying the vector by $1/\alpha$, which multiplies the magnitude by $|1/\alpha| = 1/|\alpha|$.

The magnitude of a sum is a little more complicated. There is a formula, but we're don't have the tools to discuss it yet. For now, let's just say that (1) the magnitude of the sum of two vectors is only equal to the sum of the two vectors magnitudes if the two vectors point in the same direction, and (2) in every other case, the magnitude of the sum is smaller than the sum of the magnitudes.



Graphically, this makes a lot of sense. If the two vectors are aligned, then their magnitudes add up. In every other case, $\|\vec{a}+\vec{b}\|$ represents the length of the straight path whereas $\|\vec{a}\|+\|\vec{b}\|$ is the length of the detour.

Problem 3: Fish activity – Part 1.

The google sheet linked on canvas contains the coordinates of each fish from the video every 0.33 s. Each row is a different frame. The columns are as follows: "t" is the time since the beginning of the video, "x1" is the x coordinate of the first fish, "y1" is the y coordinate of the first fish, "x2" is the x coordinate of the second fish, and "y2" is the y coordinate of the second fish. Time is measured in seconds. Coordinates are in pixel. In the questions, "row i" means your row, "row i+1" means the row below yours, "row i+2" is the row below that one, and so on.

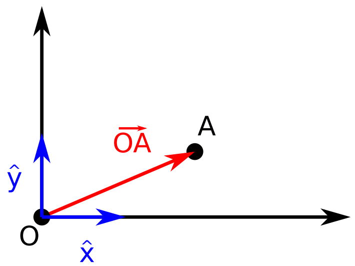
1. Compute the vector going from fish 1 to fish 2. Write its coordinates in the "x12" and "y12" columns.
2. Compute the distance between the two fish in your row. Write it in the "d12" column.
3. Compute the distance between the position of fish 1 in row i and its position in the row i+1. Write it in the "d1[i:i+1]" column.
4. Compute the distance between the position of fish 1 in row i and its position in row i+2. Write it in the "d1[i:i+2]" column.
5. Compute the distance traveled by fish 1 between row i and row i+2. Write it in the "d1t[i:i+2]" column.
   Note: Whereas in the previous question you computed the length of the straight line going from the position in row i to the position in row i+2, in this question you need to account for the fact that the fish first goes from its position in row i to its position in row i+1, then to its position in row i+2. To compute the total distance traveled, you need to compute the length of each leg of the trip (from position i to position i+1, then from position i+1 to position i+2) and add them together.

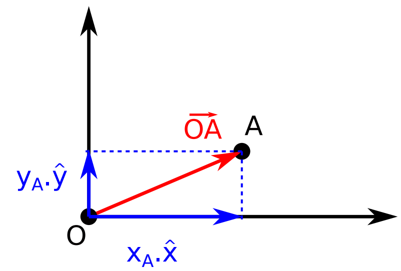
1.7 Unit vectors

A unit vector is a vector with magnitude 1. They're usually written with a hat instead of an arrow above the letter(s), e.g., $\hat{x}$ and $\hat{y}$. Any vector can be turned into a unit vector by dividing it by its own magnitude: let $\hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$, then $\|\hat{a}\| = \left\| \frac{\vec{a}}{\|\vec{a}\|} \right\| = \frac{\|\vec{a}\|}{\|\vec{a}\|} = 1$.

1.8 Basis vectors

For each axis, we can define a unit vector along that axis. Those are called basis vectors. They're usually called either $\hat{x}$ and $\hat{y}$, or $\hat{\imath}$ and $\hat{\jmath}$.





Since $\hat{x}$ has length 1, $x_A\hat{x}$ (the number x_A multiplied by the vector $\hat{x}$) corresponds to "moving along the x axis by a length x_A. Similarly, $y_A\hat{y}$ corresponds to "moving along the y axis by a length y_A". Therefore, $x_A\hat{x} + y_A\hat{y}$ is none other than $\overrightarrow{OA}$, which goes from O to A.

In terms of components, $\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (move by 1 along the x axis; don't move at all along the y axis) and $\hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (move by 1 along the y axis; don't move at all along the x axis). This is consistent with the decomposition of $\vec{a}$ we just wrote:

$$x_A\hat{x} + y_A\hat{y} = x_A \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_A \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y_A \end{bmatrix} = \begin{bmatrix} x_A + 0 \\ 0 + y_A \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \end{bmatrix} = \overrightarrow{OA}.$$

1.9 Dot product

The dot product is a type of multiplication of two vectors, however its result is a number, not a vector. There are two ways to define it. The two definitions look quite different, but they are fully equivalent.

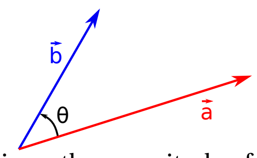
Component definition

The dot product of two vectors $\vec{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$ is the number $a_x b_x + a_y b_y$. I find the formula easiest to remember in column notation:

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \end{bmatrix} = a_x b_x + a_y b_y$$

Since $a_x b_x + a_y b_y = b_x a_x + b_y a_y$, this is a commutative product: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

Geometrical definition



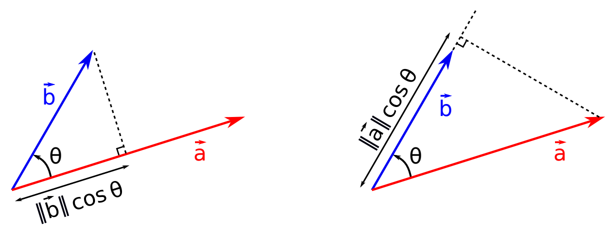
The dot product of $\vec{a}$ and $\vec{b}$ can also be defined in terms of their magnitudes and the angle θ between them:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\ \|\vec{b}\| \cos\theta$$

In words: the dot product of two vectors is equal to the magnitude of the first vector, times the magnitude of the second vector, times the angle between the two vectors.

Both magnitudes and the cosine are regular numbers (not vectors), so the right-hand-side is just a regular multiplication of three numbers. Since $\cos(-\theta) = \cos\theta$, it does not matter whether θ is measured from $\vec{a}$ to $\vec{b}$ or the other way around.

Geometrical Interpretation



On the left, we have projected the tip of $\vec{b}$ orthogonally onto $\vec{a}$. This forms a right triangle with hypothenuse $\|\vec{b}\|$. The length of the adjacent side is $\|\vec{b}\|\cos\theta$. The dot product $\vec{a}\cdot\vec{b}$ is the product of that ($\|\vec{b}\|\cos\theta$) by the magnitude of $\vec{a}$.

Alternatively, we can project $\vec{a}$ onto $\vec{b}$. This produces a second right triangle with adjacent side $\|\vec{a}\|\cos\theta$. Further multiplying by $\|\vec{b}\|$ yields the dot product.

In other words, the dot product of two vectors is equal to the magnitude of one of those vectors (either one) times the length of the projection of the other vector onto the first vector.

Dot product with a unit vector

If one of the vectors is a unit vector (let's say it's $\vec{a}$, which we'll write $\hat{a}$ to highlight that it's a unit vector), then the dot product simplifies to $\hat{a}\cdot\vec{b} = \|\vec{b}\|\cos\theta$, which is just the length of the projection of $\vec{b}$ onto $\hat{a}$.

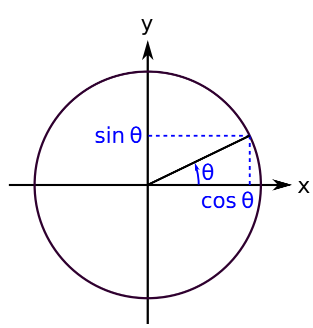
This is often a convenient way to find out how much of a vector goes in a certain direction.

Problem 4

Use the column notation to compute $\vec{a}\cdot\hat{x}$ and $\vec{a}\cdot\hat{y}$. Call a_x and a_y the components of $\vec{a}$. Explain why this is consistent with what we just discussed.

Sign of the dot product

$$\vec{a}\cdot\vec{b} = \underbrace{\|\vec{a}\|}_{\text{always} >0}\ \underbrace{\|\vec{b}\|}_{\text{always} >0}\ \underbrace{\cos\theta}_{>0 \text{ or } <0 \text{ depending on } \theta}$$



If $|\theta| < \pi/2$, then $\cos\theta > 0$.

If $\pi/2 < |\theta| < \pi$, then $\cos\theta < 0$.

Another way to look at it:

- If $\vec{a}$ and $\vec{b}$ are more aligned than anti-aligned, then $\cos\theta > 0$ and $\vec{a}\cdot\vec{b} > 0$.
- If $\vec{a}$ and $\vec{b}$ are more anti-aligned than aligned, then $\cos\theta < 0$ and $\vec{a}\cdot\vec{b} < 0$.

Problem 5: Fish activity – Part 2.

We want to find out whether fish 2 is ahead of or behind fish 1.

1. Compute the displacement vector of fish 1 between your row and the next row (the position of fish 1 in the next row relative to its position in your row). It points in direction of the front of the fish (it's not the only reasonable way to define the front, but it's the one we'll use).
2. Compute the vector going from fish 1 to fish 2.
3. Compute the dot product of the two vectors. Is fish 2 ahead or behind?
4. How much of the displacement of fish 1 between your row and the next row was done in the direction of fish 2? In other words, what is the component of the displacement vector of fish 1 along the axis going from fish 1 to fish 2?
5. Compare your last result to the distance traveled by fish 1 between those two frames (the magnitude of displacement vector you just computed). What is the meaning of their ratio?

Note 1: "Compute the vector" means compute both coordinates. The magnitude is not enough; it contains no information about the direction of the vector.
Note 2: The axis going from fish 1 to fish 2 keeps changing as the fish move. For the purpose of the last question, use the positions of the fish in your row.

Angle between two vectors

The equivalence of the two dot product definitions provides a convenient way to compute the angle between two vectors if you have their components. Say the two vectors are $\vec{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$. First, let's isolate θ in the geometrical definition of the dot product:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta \implies \theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|}\right)$$

We've already discussed how to write every term on the right-hand in terms of the components:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y \,, \quad \|\vec{a}\| = \sqrt{a_x^2 + a_y^2} \,, \quad \|\vec{b}\| = \sqrt{b_x^2 + b_y^2}$$

By plugging those expression into the expression for θ above, we get a formula for θ as a function of the components of the two vectors.

Caveat: Since $\cos(\theta) = \cos(-\theta)$, *this method cannot tell you the sign of* θ. *In other words, you don't know whether you're getting the angle between* $\vec{a}$ *and* $\vec{b}$ *or the angle between* $\vec{b}$ *and* $\vec{a}$.

Problem 6: Fish activity – Part 3.

1. Compute the displacement vector of each fish between your row and the next row.
2. Compute the angle between the headings of the two fish (in radians).
3. Are the of the fish closer to being aligned (same direction) or anti-aligned (opposite direction)? You can get that information either from the angle or from the sign of the dot product you computed on your way to computing the angle.

1.10 Notational rigor and common mistakes

When it comes to vectors, it's really important to be rigorous with your words and with your notations. Here are some common mistakes that may not seem critical at first but will cause issues down the line:

- You cannot add a number and a vector. For example, $0.5 + \begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$ is not a permitted operation. It feels like the answer should be $\begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$, but it's not. There is no right answer at all. Every seemingly reasonable way to define this addition ends up breaking the self-consistency of vector algebra. You can, however, write $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 1 \\ 2.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$. This time you're adding two vectors, which is allowed.
- You cannot take the inverse of a vector, or the square of a vector. You cannot apply mathematical functions like cos, sin, tan, exp, log, etc to a vector. Those only work on numbers. You can apply them to a number that was constructed from a vector though. For example, you cannot write $\exp\left(\overrightarrow{AB}\right)$, but you can write $\exp(AB)$, or $\exp(\|\overrightarrow{AB}\|)$, or $\exp(\overrightarrow{AB} \cdot \overrightarrow{CD})$.
- $\|AB\|$ doesn't make much sense. AB is the distance between A and B. It's a positive number. The magnitude is an operation you perform on vectors. You can write $\|\overrightarrow{AB}\|$, which is the magnitude of the vector $\overrightarrow{AB}$. You can write $AB = \|\overrightarrow{AB}\|$, which you can read as "the distance between A and B is equal to magnitude of the vector going from A to B".

  Technically you can define a magnitude operation for numbers, but it ends up being the same thing as the absolute value. A distance is a positive number, so it doesn't make much sense to take its absolute value. It's not an illegal operation, but it doesn't achieve anything.
- A point is not a vector. This is a bit confusing, because both can be written as column vectors. For example, you can write the coordinates of a point A as $A = \begin{bmatrix} x_A \\ y_A \end{bmatrix}$, and you can write the components of its position vector as $\overrightarrow{OA} = \begin{bmatrix} x_A \\ y_A \end{bmatrix}$. You can write $-\overrightarrow{OA} = \begin{bmatrix} -x_A \\ -y_A \end{bmatrix}$, but you cannot write $-A = \begin{bmatrix} -x_A \\ y_A \end{bmatrix}$. You can give a name to the mirror image of A across the origin, let's call it A', and you can write $A' = \begin{bmatrix} -x_A \\ -y_A \end{bmatrix}$, but you cannot take the negative of a point (you cannot write $A' = -A$). You cannot multiply a point by a number, add two points, or subtract two points either. You can only do those with vectors. For example you can write $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} x_B \\ y_B \end{bmatrix} - \begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix}$, but you cannot write $\overrightarrow{AB} = B - A = \begin{bmatrix} x_B \\ y_B \end{bmatrix} - \begin{bmatrix} x_A \\ y_A \end{bmatrix}$.
- The dot symbol is not reserved for the dot product. It's commonly used to denote the product of two numbers, or the product of a number and a vector. Therefore, the meaning of a dot depends on whether the nature (vector or number) or the two objects being multiplied.

  Consider four points A, B, C, D. The following expressions are all valid, but they all mean something different. Write each one in terms of the four points' coordinates.

  $AB.CD =$

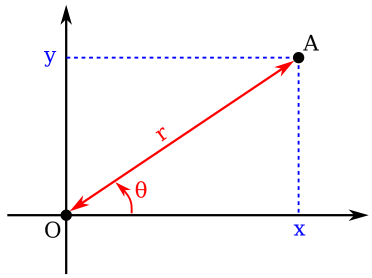
  $AB.\overrightarrow{CD} =$

  $\overrightarrow{AB}.CD =$

  $\overrightarrow{AB}.\overrightarrow{CD} =$

- You cannot compute the inverse of a vector, or raise a vector to power power (e.g., compute the square of a vector). Similarly, common mathematical functions (cos, sin, tan, exp, log, etc) only operate on numbers. For example, you cannot write $\theta = \cos\left(\begin{bmatrix} 2.1 \\ 1.4 \end{bmatrix}\right)$.

1.11 Polar coordinates



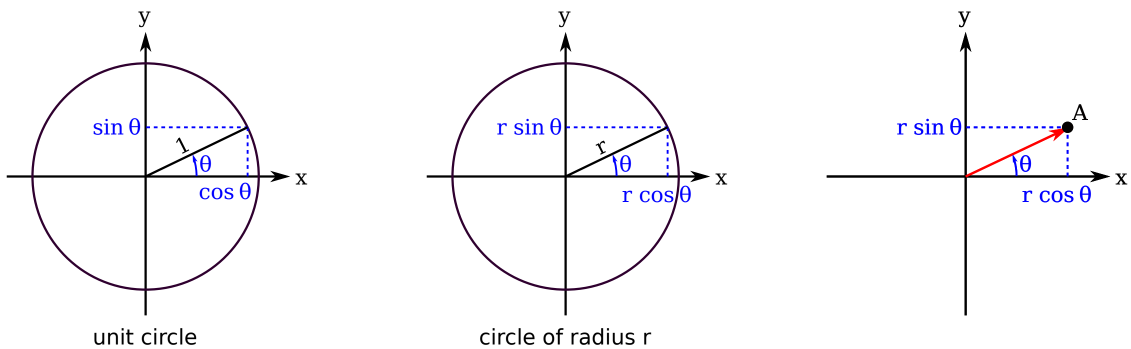
One way of communicating where a point A is located is with its cartesian coordinates x and y (I'm dropping the A subscript because there's only one point so it's clear which point I'm talking about). Another option is to give the distance r between the point and the origin and the angle θ between the positive x axis (in other words, $\hat{x}$) and the point's position vector ($\overrightarrow{OA}$). Together, r and θ are known as the polar coordinates of the point.

Comments:

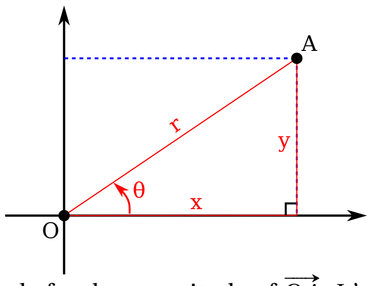
- θ must be measured from the positive half of the x axis. Measuring it from the negative half of the x axis yields a different angle ($\theta + \pi$).
- Like cartesian coordinates, polar coordinates only make sense once you've chosen an origin O and an axis x. (Cartesian coordinates also require a y axis, however you're not really free to choose it because it has to be perpendicular to the x axis and go through the origin.)

Relationship between cartesian and polar coordinates

The two sets of coordinates, cartesian and polar, contain the exact same information: the location of A. Each set of coordinates can be computed from the other. To get x and y from of r and θ, think about the trigonometric circle:



The formulas are $x = r\cos\theta$ and $y = r\sin\theta$.



To get r and θ from of x and y, think about the right triangle formed by O, A, and the projection of A onto the x axis. The Pythagorean theorem gives

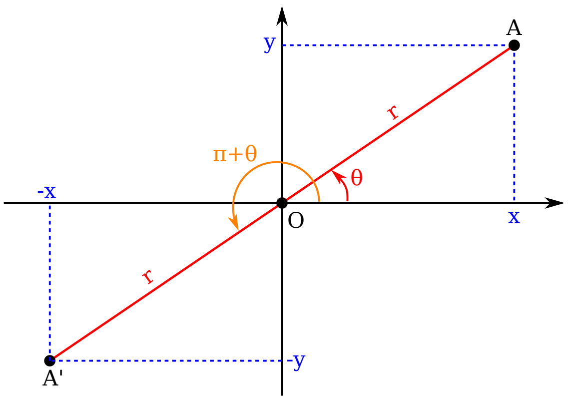
$$r = \sqrt{x^2 + y^2}.$$

Note that r is the same thing as OA or $\|\overrightarrow{OA}\|$. We could also have used the formula for the magnitude of $\overrightarrow{OA}$. It's really the same thing though: if you think back to how we got the magnitude formula, it was by using the exact same triangle.

To get θ, we use the definition of the tangent and solve for θ:

$$\tan\theta = \frac{y}{x} \quad \Longrightarrow \quad \theta = \begin{cases} \arctan(y/x) & x > 0 \\ \pi + \arctan(y/x) & x < 0 \end{cases}$$

The $\pi + \ldots$ comes from the fact that $\arctan(y/x)$ cannot tell the difference between A and its mirror image across the origin because $-y/-x = x/y$. The arctan function always gives an angle between $-\pi/2$ and $\pi/2$, corresponding to a point with $x > 0$. If the actual point has $x < 0$, then we get its mirror image instead. We fix that by adding π to θ, which corresponds to a rotation of π around the origin, i.e., unmirroring the mirror image:



Problem 7: Fish activity – Part 4.

The center of the tank is located at $\begin{bmatrix} 579.743 \\ 532.006 \end{bmatrix}$. The radius of the tank is 455.476. Both are in pixels.

1. Compute the polar coordinates of fish 1 using the center of the tank as the origin.
2. Compute the distance between fish 1 and the nearest wall.

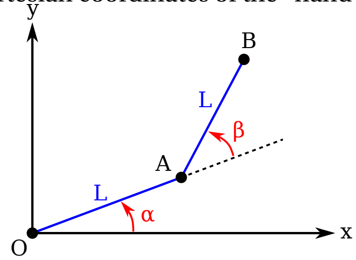
Problem 8: Diagonal unit vector.

In this problem, we find the components of the unit vector pointing along the diagonal line $y = x$ in the up-and-right direction twice, first without using polar coordinates, then using polar coordinates.

1. Draw a sketch to show that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the right direction. It's not a unit vector though. Make a unit vector out of it. Double check that the magnitude of the result is 1.
2. What is the angle between the x axis and the unit vector we're after? Use the polar-to-cartesian formulas (or the trigonometric circle) to write the components of the unit vector in terms of the angle. After simplification you should get the same result as in question 1.

Problem 9: Robotic arm.

In articulated robotic arms, you don't usually control the position of the "hand" directly. Instead, you control the motors that control the angle of each joint. In order have the arm perform a specific motion in space (grab an object, draw something, etc), it is often necessary to compute the relationship between those joint angles and the cartesian coordinates of the "hand".



Here the arm has two segments with the same length L, and we control the angles α and β. The dashed line represents the direction of the first arm segment. We want the coordinates of B (x_B and y_B) as a function of L, α, and β.

1. Compute x_A and y_A (cartesian coordinates of A).
2. Compute the angle between $\hat{x}$ and $\overrightarrow{AB}$.
3. Compute the components of $\overrightarrow{AB}$.
4. Compute x_B and y_B (cartesian coordinates of B).

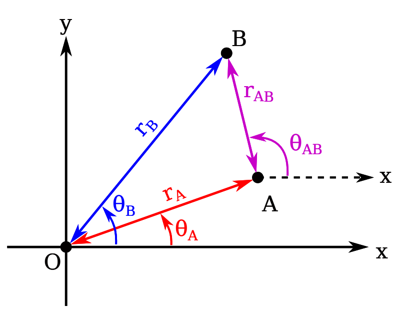
Note: The relationship we obtained answers questions like "if I set the joint angles to those values, where will the hand be in space?". For the purpose of controlling the arm, though, the more interesting question is the inverse one: "if I want the hand to get to this location in space, what angle do I need at each joint". That is answered by inverting the relationship we found, i.e., solving for α and β in terms of L, x, and y. It's more difficult, and not quite worth the trouble in this class. If you want to learn more about this type of problem, look up "inverse kinematics".

1.12 Polar basis

A limitation of polar coordinates

Polar coordinates are not well suited for vector arithmetic. Say we're interested in the relative position vector $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. If we have the cartesian coordinates of A and B, we can subtract them from each other to get the cartesian components of the relative position: $\overrightarrow{AB} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix}$.

That does **not** work with polar coordinates, at least not in general:



$$r_{AB} \neq r_B - r_A$$
$$\theta_{AB} \neq \theta_B - \theta_A$$

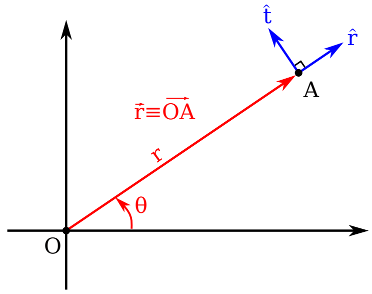
In fact, the simplest way to compute the polar coordinates of $\overrightarrow{AB}$ from the polar coordinates of A and B is to first compute the cartesian coordinates of A and B, then perform the subtraction in cartesian coordinates, then convert the result back to polar coordinates.

Problem 10: Relative position in polar coordinates.

Do it: compute the polar coordinates of $\overrightarrow{AB}$ (r_{AB} and θ_{AB}) as a function of the polar coordinates of A and B (r_A, θ_A, r_B, and θ_B).

Polar basis

The polar basis is a way to address this issue. Say we're studying the motion of a point A, and that motion is circular, so we decide to describe its position in polar coordinates. For other vectors (anything besides the position vector: displacement vectors, velocity vector, force vectors, etc), we use a special cartesian basis that rotates with the moving point A:



$\vec{r}$ is an alternate notation for the position vector $\overrightarrow{OA}$. If there were multiple points, say A and B, we could call their position vectors $\vec{r}_A$ and $\vec{r}_B$. In there is only one point, or if there are multiple points but we're primarily interested in the motion of one of those points, then we often drop the subscript and use the more compact $\vec{r}$ notation. Since the first polar coordinate, r, is the magnitude of the position vector, you can then write $r = \|\vec{r}\|$, which is reminiscent of $OA = \|\overrightarrow{OA}\|$.

$\hat{r}$ is called the *radial unit vector*. It is the unit vector constructed from $\vec{r}$, i.e., it has the same direction as $\vec{r}$ but its magnitude is 1: $\hat{r} = \frac{\vec{r}}{\|\vec{r}\|}$. You can also write that as $\hat{r} = \frac{\vec{r}}{r}$, or $\hat{r} = \widehat{OA} = \frac{\overrightarrow{OA}}{\|\overrightarrow{OA}\|}$. Using the the polar-to-cartesian formulas (or better thinking back to the trigonometric circle which allowed us to obtain those formulas in the first place), we can write $\hat{r} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

$\hat{t}$ is called the *tangential unit vector*. It is obtained by rotating $\hat{r}$ by $\pi/2$ counterclockwise, the same way you would construct $\hat{y}$ from $\hat{x}$. The point is that $(\hat{r}, \hat{t})$ is a cartesian basis just like $(\hat{x}, \hat{y})$ is a cartesian basis. In cartesian coordinates $\hat{t} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

Just like any vector can be decomposed into its x and y components, any vector can be decomposed into its radial and tangential components. One way to obtain them is to take the dot product of the vector with $\hat{r}$ (for the radial component) and/or $vecut$ (for the tangential component).

Problem 11: Constructing a cartesian basis.

1. Check that $\hat{r}$ and $\hat{t}$ are unit vectors.
2. Check that $\hat{r}$ and $\hat{t}$ are perpendicular to each other.
3. More generally, show that for any vector $\vec{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$, the vectors $\begin{bmatrix} a_y \\ -a_x \end{bmatrix}$ and $\begin{bmatrix} -a_y \\ a_x \end{bmatrix}$ both have the same magnitude as $\vec{a}$ and both are perpendicular to $\vec{a}$.
   Note: The first option corresponds to rotating $\vec{a}$ by $+\pi/2$. The second option corresponds to rotating $\vec{a}$ by $-\pi/2$. Unless specified otherwise, the second vector of a cartesian basis is obtained by rotating the first vector by $+\pi/2$. The most common case, with $\hat{x}$ pointing to the right and $\hat{y}$ pointing up, follows this convention.
4. Construct a cartesian basis whose first vector is along $\vec{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$. Do not assume $\vec{a}$ is a unit vector. Write the x and y components of the two basis vectors in terms of a_x and a_y.

Problem 12: Fish activity – Part 5.

1. Compute the polar components (radial and tangential) of the displacement of fish 1 between your row and the next row. Use the polar coordinate system centered on the center of the tank, with fish 1 as the "point of interest" (the one that defines r and θ).
2. Describe, in plain English, the meaning of the two components you just computed.

2 Units and Dimensional Analysis

Relevant sections in University Physics Volume 1: 1.2, 1.3, 1.4.

2.1 The Need for Units

Most numerical values in physics are meaningless unless they include a unit.

Example: "I live 5 from campus". Five what? Miles? Inches? Blocks? Minutes? "Five" is meaningless on its own. It only becomes meaningful if you specify a unit.

A notable exception is the ratio of two quantities that have the same unit. For example, $\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypothenuse}}$. As long as we use the same unit for both lengths, the result does not depend on which unit we chose. Say the lengths are 1 ft and 2 ft, then the cosine is $\frac{1\,\text{ft}}{2\,\text{ft}} = 0.5$. If we write the lengths in inches instead, we get the same value: $\frac{1\,\text{ft}}{2\,\text{ft}} = \frac{12\,\text{in}}{24\,\text{in}} = 0.5$. The point is, we don't need to specify a unit when reporting the value of the cosine because it has the same value regardless of which units we used to compute it (as long as we were consistent and used the same unit for all lengths). The same idea applies to any quantity that is the ratio of two quantities of the same type, e.g., a length divided by a length, a time divided by a time, etc.

2.2 International Unit System

Different cultures use different units to measure the same quantities. In order to avoid ambiguity when sharing results with people from across the world, it is useful to have an international standard: a reference set of units recognized by scientists across the world. That's the *International Unit System*, or *SI system* for short. The units in that system are known as *SI units*. If I ask you to give an answer "in SI units", I mean using the units from the SI system.

The three most important SI units in this course are:

- The *meter* (symbol "m"), used to measure lengths.
- The *second* (symbol "s"), used to measure times or durations.
- The *kilogram* (symbol "kg"), used to measure masses.

As it turns out, every other unit we'll need in this course can be constructed from those three. More on that in section 2.5.

The SI system also provides a way to create smaller or larger unit of the same type using prefixes. For example, a kilogram is 10^3 grams, and a milligram is 10^{-3} gram. What's nice about those prefixes is that they can be appended to any unit: a millimeter is 10^{-3} meters, a millisecond is 10^{-3} seconds, etc. There are more details about this in section 1.2 of University Physics Volume 1. In practice, we'll only use a small subset of the prefixes, mostly micro-, milli-, centi-, kilo-, and mega-.

2.3 Units Algebra

- You can only add numerical values together if they have the same unit. When they do, that unit is also the unit of the result.

  Examples:

  - $1\,\text{m} + 2\,\text{m}$ is allowed because both values are in meters. The result is $3\,\text{m}$, also in meters. One way to think about it is to say that we factored out the *m*.
  - $1\,\text{m} + 2\,\text{cm}$ cannot be computed directly because the units (m and cm) are different, however you can use unit conversion to bring them to the same unit then add them: $1\,\text{m} + 2\,\text{cm} = 1\,\text{m} + 0.02\,\text{m} = 1.02\,\text{m}$, or $1\,\text{m} + 2\,\text{cm} = 100\,\text{cm} + 2\,\text{cm} = 102\,\text{cm}$.

- $1\,\text{m} + 1\,\text{s}$ is not allowed at all. The two units (m and s) are not just different, there is no way to convert one into the other, so there is no way to perform the addition at all. If an addition of this type shows up in your work, you must have made a mistake in a previous step.

- When multiplying, dividing, or taking powers, you can treat the units the same way you would any other unknown literal, except at the end you consolidate all the units together and write them after the numerical value.

  Examples:

  - $1\,\text{m} \times 2\,\text{m} = (1 \times 2) \times (\text{m} \times \text{m}) = 1\,\text{m}^2$.
  - $\dfrac{3\,\text{m}}{2\,\text{s}} = \dfrac{3}{2}\dfrac{\text{m}}{\text{s}} = 1.5\,\text{m/s}$.
  - $\sqrt{4\,\text{m}^2} = (\sqrt{4})(\sqrt{\text{m}^2} = 2\,\text{m}$.
  - $\dfrac{1}{4\,\text{m}} = \left(\dfrac{1}{4}\right)\left(\dfrac{1}{\text{m}}\right) = 0.25\,\text{m}^{-1}$.
  - $(3\,\text{m})^3 = (3)^3 \times (\text{m})^2 = 27\,\text{m}^3$.

Note: There are two common ways to write the inverse of a unit: with a division bar or a negative exponent. For example, meters per second can be written either m/s or $\text{m}\,\text{s}^{-1}$. The main issue with the division bar is that it can become ambiguous when there are more than two units. For example, $\text{m/s}\,\text{kg}$ (meters per second kilogram) could mean either $(\text{m/s})\text{kg}$ or $\text{m}/(\text{s}\,\text{kg})$, which are two different units. You can lift that ambiguity by using parentheses ($(\text{m/s})\text{kg}$ vs $\text{m}/(\text{s}\,\text{kg})$) or a fraction bar ($\dfrac{\text{m}\,\text{kg}}{\text{s}}$ vs $\dfrac{\text{m}}{\text{s}\,\text{kg}}$). Or you can use the exponent notation, which bypasses the ambiguity altogether: $(\text{m/s})\,\text{kg}$ is written $\text{m}\,\text{s}^{-1}\,\text{kg}$, whereas $\text{m}/(\text{s}\,\text{kg})$ is written $\text{m}\,\text{s}^{-1}\,\text{kg}^{-1}$. I tend to use the latter, although I sometimes use parentheses as well.

2.4 Unit conversion

Unit conversions can also be handled by thinking of the units as unknown literals. If we know the relationship between two units (for example, $1\,\text{cm} = 0.01\,\text{m}$), we can plug it into any expression to replace one unit with the other.

Examples:

- $3\,\text{cm} = 3 \times (0.01\,\text{m}) = (3 \times 0.01)\text{m} = 0.03\,\text{m}$.
- $3\,\text{cm}^2 = 3 \times (0.01\,\text{m})^2 = 3 \times (0.01)^2 \times (\text{m})^2 = 0.0003\,\text{m}^2$.
- $3\,\text{mph} = 3 \times \left(\dfrac{1\,\text{mile}}{1\,\text{hour}}\right) = 3 \times \left(\dfrac{1609\,\text{m}}{3600\,\text{s}}\right) = \dfrac{3 \times 1609}{3600}\text{ms}^{-1} \approx 1.34\,\text{m/s}$

Note: The book suggests a slightly different but ultimately equivalent strategy. Check it out here: University Physics Volume 1, section 1.3. Use whichever one works best for you.

Problem 13: Unit conversion.

See problems 39 and 42 at the end of University Physics Volume 1, chapter 1.

2.5 Base Units vs Derived Units

What's special about meters, seconds, and kilograms is that every other unit we'll encounter in mechanics can be written as a combination of those three. More specifically, every other SI unit can be written as $\text{m}^a\,\text{s}^b\,\text{kg}^c$ where a, b, c are three numbers. For example, meters per second correspond to $a = 1$, $b = -1$, $c = 0$.

For this reason, meters, seconds, and kilograms are called "base units" whereas other mechanical units are called "derived units".

Note: I specified "in mechanics" and "mechanical units" because there are other base units beyond meters, seconds, and kilograms, but they don't come up in mechanics (in this course anyway).

2.6 Types of quantities

We've said you can only add two numerical values if they have the same unit. What about two unknown literals, say a and b? Their units, like their numerical values, are yet unspecified. Still, we can ask: could a and b, at least in principle, be converted to the same unit? For example, the radius of the Earth and the size of a bacteria are typically given in different units, e.g., kilometers and microns, but they're both lengths, so they can both be converted to a common length unit (whichever one; the SI unit of length, the meter, would be a natural choice). Therefore, we can add them. If a is the radius of the Earth and b is the size of a bacteria, then $a+b$ is a meaningful expression to write. Going back to the general case, we can only add two quantities a and b if they are of the same type.

For each SI unit, we can define a corresponding *type of quantity* as the set of all quantities that can be converted to that unit. For example, anything that can be converted to meters is of the type *length* (symbol L). Any quantity that can be converted to seconds is a *time* (symbol T). Any quantity that can converted to kilograms is a *mass* (symbol M).

The same way all SI units in mechanics can be written as a combination of meters, seconds, and kilograms, the type of every quantity we encounter in mechanics can be written as a combination of length, time, and mass. For example, a speed is a length divided by a time.

Notations:

- The type of a quantity is denoted by putting square brackets around the name of the quantity. For example: [speed]=[length]/[time]. We're not talking about any specific speed, or length, or time here. We're making a much more general statement about the fact that any speed one may possibly compute is of type [length]/[time], i.e., can be converted to m/s. We say that *a speed has the dimension of a length divided by a time.*
- Since length, time, and mass play a special role among types of quantities, there were given their own symbols: L for length, T for time, and M for mass. Using those symbols, rewrite that [speed]=[length]/[time] as [speed]=$L.T^{-1}$.

2.7 Dimensional analysis

Dimensional analysis refers to the analysis of the types of quantities involved in a formula. It uses the same algebraic rules as units:

- You can only add two quantities if they are of the same type.
- In most other regards, treat quantity types as unknown literals. In particular, you can multiply, divide, take powers, and simplify them like unknown literals.

Dimensional analysis of a single quantity

Performing a dimensional analysis of a quantity, also called finding the dimension of the quantity, means to write it in terms of the base quantities (length L, time T, mass M). For example, to determine the dimension of a speed, we need a formula that relates speed to quantities whose types we already know. A speed is a distance traveled divided by a time elapsed, therefore we can write
[speed] = [distance traveled]/[time elapsed] = $L/T = LT^{-1}$. The final result, [speed]=$L\,T^{-1}$, is true of any speed in any context.

Problem 14: Dimension of common mechanics quantities.

1. Use any surface area formula to find the dimension of an area.
2. Use any volume formula to find the dimension of a volume.
3. Use the formula for the average velocity (from the kinematics lecture notes) to find the dimension of a velocity.
4. Use the formula for the average acceleration to find the dimension of an acceleration.

5. Use the famous formula $E = mc^2$, where E is the energy of a resting object, m is the mass of the object, and c is the speed of light, to find the dimension of an energy.

Dimensional analysis as a consistency check

Dimensional analysis can be very helpful to catch mistakes in formulas and equations. The idea is that any addition in the formula should involve quantities of the same type (i.e., every term should have the same dimension), and the two sides of an equation should always be of the same type (i.e., the two sides should have the same dimension).

Problem 15: Checking a formula by dimensional analysis.

See problems 50 and 80 at the end of University Physics Volume 1, chapter 1.

Problem 16: Checking free fall formulas.

In the formulas below, h is a height, v_0 is a speed, and g is the acceleration of gravity. Some formulas are dimensionally consistent, some are not. Perform a dimensional analysis of each formula. State whether they are dimensionally consistent or not. If they are, make up a question they could be a plausible answer to (based on dimensional analysis and which variables make up the formula). If they are not, give one way to change the formula to make it dimensionally consistent.

1. $\sqrt{gh}$.
2. $\sqrt{\frac{h}{g}}$.
3. $v_0 + 2gh$.